Harry Hatchard

<u>Easter Eggs</u>

1.

a.

$$f(x,y,z) = zx - xy\sin(xyz) + y\cos(xyz) = 0$$

$$\left(\frac{\partial x}{\partial y}\right)_z = -\frac{\left(\frac{\partial f}{\partial y}\right)_{x,z}}{\left(\frac{\partial f}{\partial x}\right)_{y,z}} = -\frac{-x(\sin(xyz) + xyz\cos(xyz)) + \cos(xyz) - xyz\sin(xyz)}{z - y(\sin(xyz) + xyz\cos(xyz)) - y^2z\sin(xyz)}$$

$$= \frac{x\sin(xyz) + x^2yz\cos(xyz) - \cos(xyz) + xyz\sin(xyz)}{z - y\sin(xyz) - xy^2z\cos(xyz) - y^2z\sin(xyz)}$$

$$= \frac{(yz+1)x\sin(xyz) + (x^2yz - 1)\cos(xyz)}{z - (yz+1)y\sin(xyz) - xy^2z\cos(xyz)}$$

$$\left(\frac{\partial y}{\partial z}\right)_z = -\frac{\left(\frac{\partial f}{\partial z}\right)_{x,y}}{\left(\frac{\partial f}{\partial y}\right)_{x,z}} = -\frac{x - x^2y^2\cos(xyz) - xy^2\sin(xyz)}{-x(\sin(xyz) + xyz\cos(xyz)) + \cos(xyz) - xyz\sin(xyz)}$$

$$= \frac{x - xy^2\sin(xyz) - x^2y^2\cos(xyz)}{(yz+1)x\sin(xyz) + (x^2yz - 1)\cos(xyz)}$$

Alternative method

$$zx - xy\sin xyz + y\cos xyz = 0$$

$$z\left(\frac{\partial x}{\partial y}\right)_z - \left(\left(\frac{\partial x}{\partial y}\right)_z y + x\right)\sin xyz - xyz\cos xyz\left(\left(\frac{\partial x}{\partial y}\right)_z y + x\right) + \cos xyz$$
$$- yz\sin xyz\left(\left(\frac{\partial x}{\partial y}\right)_z y + x\right)$$

$$= \left(\frac{\partial x}{\partial y}\right)_z (z - y\sin xyz - xy^2z\cos xyz - y^2z\sin xyz)$$
$$+ (-x\sin xyz - x^2yz\cos xyz + \cos xyz - xyz\sin xyz)$$

$$\rightarrow \left(\frac{\partial x}{\partial y}\right)_z = \frac{x\sin xyz + x^2yz\cos xyz - \cos xyz + xyz\sin xyz}{z - y\sin xyz - xy^2z\cos xyz - y^2z\sin xyz}$$

$$= \frac{(yz+1)x\sin xyz + (x^2yz - 1)\cos xyz}{z - (yz+1)y\sin xyz - xy^2z\cos xyz}$$

$$x - x\left(\frac{\partial y}{\partial z}\right)_x \sin xyz - x^2y\cos xyz\left(\left(\frac{\partial y}{\partial z}\right)_x z + y\right) + \left(\frac{\partial y}{\partial z}\right)_x \cos xyz - yx\sin xyz\left(\left(\frac{\partial y}{\partial z}\right)_x z + y\right)$$

$$= \left(\frac{\partial y}{\partial z}\right)_x (-x\sin xyz - x^2yz\cos xyz + \cos xyz - xyz\sin xyz) + x - x^2y^2\cos xyz - y^2x\sin xyz$$

$$\rightarrow \left(\frac{\partial y}{\partial z}\right)_x = \frac{x - x^2y^2\cos xyz - xy^2\sin xyz}{x\sin xyz + x^2yz\cos xyz - \cos xyz + xyz\sin xyz}$$

$$= \frac{x - xy^2\sin xyz - x^2y^2\cos xyz}{(yz+1)x\sin xyz + (x^2yz - 1)\cos xyz}$$

b.

$$x = r\cosh t \quad \rightarrow \quad \left(\frac{\partial x}{\partial r}\right)_t = \cosh t = \frac{x}{r}, \quad \left(\frac{\partial x}{\partial t}\right)_r = r\sinh t = y$$

$$y = r\sinh t \quad \rightarrow \quad \left(\frac{\partial y}{\partial r}\right)_t = \sinh t = \frac{y}{r}, \quad \left(\frac{\partial y}{\partial t}\right)_r = r\cosh t = x$$

$$\left(\frac{\partial f}{\partial r}\right)_t = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial r}\right)_t + \left(\frac{\partial f}{\partial y}\right)_y \left(\frac{\partial y}{\partial r}\right)_t, \quad \left(\frac{\partial f}{\partial t}\right)_r = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial t}\right)_r + \left(\frac{\partial f}{\partial y}\right)_y \left(\frac{\partial y}{\partial t}\right)_r$$

$$\rightarrow \quad \left(\frac{\partial f}{\partial r}\right)_t = \frac{x}{r}\left(\frac{\partial f}{\partial x}\right)_y + \frac{y}{r}\left(\frac{\partial f}{\partial y}\right)_y, \quad \left(\frac{\partial f}{\partial t}\right)_r = y\left(\frac{\partial f}{\partial x}\right)_y + x\left(\frac{\partial f}{\partial y}\right)_y$$

$$\left(\frac{\partial f}{\partial r}\right)_t^2 - \frac{1}{r^2}\left(\frac{\partial f}{\partial t}\right)_r = \left(\left(\frac{\partial f}{\partial r}\right)_t + \frac{1}{r}\left(\frac{\partial f}{\partial t}\right)_r\right)\left(\left(\frac{\partial f}{\partial r}\right)_t - \frac{1}{r}\left(\frac{\partial f}{\partial t}\right)_r\right)$$

$$= \left(\frac{x}{r}f_x + \frac{y}{r}f_y + \frac{1}{r}(yf_x + xf_y)\right)\left(\frac{x}{r}f_x + \frac{y}{r}f_y - \frac{1}{r}(yf_x + xf_y)\right)$$

$$= \left(\left(\frac{x}{r} + \frac{y}{r}\right)f_x + \left(\frac{x}{r} + \frac{y}{r}\right)f_y\right)\left(\left(\frac{x}{r} - \frac{y}{r}\right)f_x - \left(\frac{x}{r} - \frac{y}{r}\right)f_y\right)$$

$$= \frac{1}{r^2}(x+y)(x-y)(f_x + f_y)(f_x - f_y)$$

$$= \frac{x^2 - y^2}{r^2}(f_x^2 - f_y^2) = \frac{r^2\cosh^2 t - r^2\sinh^2 t}{r^2} = \frac{r^2}{r^2}(f_x^2 - f_y^2) = \left(\frac{\partial f}{\partial x}\right)_y^2 - \left(\frac{\partial f}{\partial y}\right)_x^2$$

2.

a.

i.

$$z = -1 - i \quad \rightarrow \quad |z| = \sqrt{2}, \quad |z|^2 = 2 \quad \rightarrow \quad \mathrm{Re}(|z|^2) = 2, \quad Im(|z^2|) = 0$$

$$\mathrm{Arg}(z) = -\frac{3\pi}{4}, \quad \hat{z} = e^{-\frac{3\pi}{8}i}, \quad u_1 = -1, \quad u_2 = 1$$

$$z_1^{\frac{1}{2}} = -\sqrt{2}^{\frac{1}{2}}e^{-\frac{3\pi i}{8}} = 2^{\frac{1}{4}}\cos\left(\frac{3\pi}{8}\right) - 2^{\frac{1}{4}}i\sin\left(\frac{3\pi}{8}\right), \quad z_2^{\frac{1}{2}} = -2^{\frac{1}{4}}\cos\left(\frac{3\pi}{8}\right) + 2^{\frac{1}{4}}i\sin\left(\frac{3\pi}{8}\right)$$

$$\mathrm{Re}\left(z_1^{\frac{1}{2}}\right) = 2^{\frac{1}{4}}\cos\left(\frac{3\pi}{8}\right), \quad \mathrm{Im}\left(z_1^{\frac{1}{2}}\right) = -2^{\frac{1}{4}}\sin\left(\frac{3\pi}{8}\right)$$

$$\mathrm{Re}\left(z_2^{\frac{1}{2}}\right) = -2^{\frac{1}{4}}\cos\left(\frac{3\pi}{8}\right), \quad \mathrm{Im}\left(z_2^{\frac{1}{2}}\right) = 2^{\frac{1}{4}}\sin\left(\frac{3\pi}{8}\right)$$

ii.

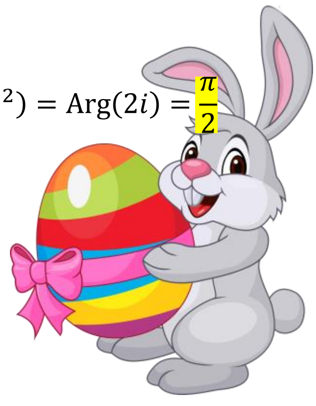
$$||z|^2| = |2| = 2, \quad \mathrm{Arg}(|z|^2) = 0$$

$$z^2 = (-1-i)(-1-i) = 1 + 2i - 1 = 2i, \quad |z^2| = |2i| = 2, \quad \mathrm{Arg}(z^2) = \mathrm{Arg}(2i) = \frac{\pi}{2}$$

$$-iz = i - 1, \quad |-iz| = \sqrt{2}, \quad \mathrm{Arg}(-iz) = \frac{3\pi}{4}$$

b.

let $z = 1 + \cos 5\theta + i\sin 5\theta, \quad z^* = 1 + \cos 5\theta - i\sin 5\theta$



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$$\frac{z}{z^*} = \frac{|z|e^{i\phi}}{|z|e^{-i\phi}} = e^{2i\phi}, \quad \tan\phi = \frac{\sin 5\theta}{1+\cos 5\theta} = \frac{2\sin\frac{5\theta}{2}\cos\frac{5\theta}{2}}{1 + 2\cos^2\frac{5\theta}{2} - 1} = \frac{\sin\frac{5\theta}{2}}{\cos\frac{5\theta}{2}} = \tan\frac{5\theta}{2}$$

$$\rightarrow \quad \phi = \frac{5\theta}{2}, \quad \frac{z}{z^*} = e^{2i\phi} = e^{5i\theta} = \cos 5\theta + i\sin 5\theta$$

3.

a.

$$\arg(z_1) = \operatorname{atan}\frac{1}{2} + 2n\pi, \quad \arg(z_2) = \operatorname{atan}\frac{1}{3} + 2n\pi$$

b.

$$\arg(z_1) + \arg(z_2) = \arg(z_1 z_2) = \arg\big((2+i)(3+i)\big) = \arg(5+5i) = \frac{\pi}{4} + 2n\pi$$

$$\rightarrow \quad \frac{\pi}{4} = \operatorname{atan}\frac{1}{2} + \operatorname{atan}\frac{1}{3}$$

c.

$$\frac{d}{dx}\operatorname{atan} x = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$$

$$\rightarrow \quad \operatorname{atan} x = \int 1 - x^2 + x^4 - \frac{x^6}{2} + \cdots dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

$$\pi \approx 4\left(\frac{1}{2} - \frac{1}{3}\left(\frac{1}{2}\right)^3 + \frac{1}{3} - \frac{1}{3}\left(\frac{1}{3}\right)^3\right) = \frac{502}{162} = 3.1173$$

4.

a.

$$\frac{dy}{dx} + \frac{1}{x}y = 1 \quad \rightarrow \quad \text{linear}, \quad \mu = e^{\int \frac{1}{x}dx} = e^{\ln x} = x$$

$$x\frac{dy}{dx} + y = x, \quad \frac{d}{dx}(xy) = x \quad \rightarrow \quad xy = \int x\,dx = \frac{x^2}{2} + C \quad \rightarrow \quad y = \frac{x}{2} + \frac{C}{x}$$

b.

$$\frac{dy}{dx} = \frac{x-y}{x} = \frac{f(x,y)}{g(x,y)} = \frac{nx - ny}{nx} = \frac{f(nx,ny)}{g(nx,ny)} \quad \rightarrow \quad \text{homogeneous}$$

$$y = vx \quad \rightarrow \quad \frac{dy}{dx} = v + x\frac{dv}{dx} = 1 - \frac{vx}{x} = 1 - v \quad \rightarrow \quad x\frac{dv}{dx} = 1 - 2v$$

$$\frac{1}{1-2v}dv = \frac{1}{x}dx \quad \rightarrow \quad \int \frac{1}{1-2v}dv = \ln x = \frac{-\ln|1-2v|}{2} = \ln\left|\frac{C}{\sqrt{1-2v}}\right|$$

$$x = \frac{C}{\sqrt{1-2v}} \quad \rightarrow \quad \frac{C}{x^2} = 1 - 2v = 1 - \frac{2y}{x} \quad \rightarrow \quad y = \frac{x}{2} - \frac{C}{2x} = \frac{x}{2} + \frac{C}{x}$$

c.

$$x\frac{dy}{dx} = x - y \quad \rightarrow \quad x\,dy = (x-y)dx \quad \rightarrow \quad (x-y)dx - x\,dy = 0$$

$$\left(\frac{\partial(x-y)}{\partial y}\right)_x = -1, \quad \left(\frac{\partial(-x)}{\partial x}\right)_y = -1 \quad \rightarrow \quad \text{exact}$$

$$f(x,y) = \int x - y\,dx + \psi(y) = \frac{x^2}{2} - yx + \psi(y)$$

$$f(x,y) = \int -x\,dy + \phi(x) = -xy + \phi(x)$$

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$$\rightarrow \quad \phi(x) = \frac{x^2}{2}, \quad \psi(y) = 0, \quad \rightarrow \quad f(x,y) = \frac{x^2}{2} - xy = \int 0\, df = C$$

$$xy = \frac{x^2}{2} + C \quad \rightarrow \quad y = \frac{x}{2} + \frac{C}{x}$$

5.

$$\frac{dy}{dx} = -y \tanh x \quad \rightarrow \quad \int -\frac{1}{y} dy = \int \tanh x\, dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$let\ u = e^x + e^{-x}, \quad du = e^x - e^{-x} dx$$

$$-\ln y = \int \frac{1}{u} du = \ln u + \ln C = \ln(C \cosh x) \quad \rightarrow \quad y = \frac{C}{\cosh x}$$

6.

$$\frac{dy}{dx} = \frac{z+2}{z-4} = \frac{1}{2}\frac{dz}{dx} - \frac{1}{2} \quad \rightarrow \quad \frac{dz}{dx} = \frac{2z+4}{z-4} + 1 = \frac{2z+4+z-4}{z-4} = \frac{3z}{z-4}$$

$$\frac{dx}{dz} = \frac{z-4}{3z} \quad \rightarrow \quad x = \int \frac{z-4}{3z} dz = \frac{z}{3} - \frac{4}{3}\ln z + C$$

$$x = \frac{x}{3} + \frac{2y}{3} - \frac{4}{3}\ln(x+2y) + C \quad \rightarrow \quad 2y - 4\ln(x+2y) - 2x + C = 0$$

7.

$$\frac{d^2u}{ds^2} + 4\frac{du}{ds} + 5u = 0, \quad \lambda^2 + 4\lambda + 5 = 0 \quad \rightarrow \quad \lambda = -2 \pm \frac{\sqrt{16-20}}{2} = -2 \pm i$$

$$u = Ae^{(-2+i)s} + Be^{(-2-i)s} = Ae^{-2s}(\cos s + i \sin s) + Be^{-2s}(\cos s - i \sin s)$$

$$= e^{-2s}(A+B)\cos s + e^{-2s}(A-B)\sin s = e^{-2s}(C_1 \cos s + C_2 \sin s), \quad C_1, C_2 \in \mathbb{C}$$

$$u(0) = e^{-2\times 0}(C_1 \cos 0 + C_2 \sin 0) = C_1 = 0$$

$$\frac{du}{ds} = -2e^{-2s}(C_1 \cos s + C_2 \sin s) + e^{-2s}(-C_1 \sin s + C_2 \cos s)$$

$$= e^{-2s}((C_2 - 2C_1)\cos s - (C_1 + 2C_2)\sin s)$$

$$u'(0) = e^{-2\times 0}\big((C_2 - 2C_1)\cos 0 - (C_1 + 2C_2)\sin 0\big) = C_2 - 2C_1 = C_2 = 2$$

$$\rightarrow \quad u(s) = 2e^{-2s}\sin s$$

8.

$$\frac{d^2x}{dt^2} + 9x = 3 - 3\pi t \quad \rightarrow \quad \lambda^2 + 9 = 0 \quad \rightarrow \quad \lambda = \pm 3i$$

$$Ae^{3it} + Be^{-3it} = A\cos 3t + Ai\sin 3t + B\cos 3t - Bi\sin 3t = (A+B)\cos 3t + (A-B)i\sin 3t$$

$$= C_1 \cos 3t + C_2 \sin 3t$$

$$x_a(t) = C_3 t + C_4, \quad x_a'(t) = C_3, \quad x_a''(t) = 0, \quad 9(C_3 t + C_4) = 3 - 3\pi t$$

$$\rightarrow \quad C_3 = -\frac{\pi}{3}, \quad C_4 = \frac{1}{3}$$

$$x(t) = C_1\cos 3t + C_2 \sin 3t - \frac{\pi t}{3} + \frac{1}{3}$$

$$x(0) = C_1 + \frac{1}{3} = \frac{1}{3} \quad \rightarrow \quad C_1 = 0, \qquad x'\left(\frac{\pi}{3}\right) = 3C_2\cos\pi - \frac{\pi^2}{9} = 0 \quad \rightarrow \quad C_2 = \frac{\pi^2}{27}$$

$$x(t) = \frac{\pi^2}{27}\sin 3t - \frac{\pi t}{3} + \frac{1}{3}$$

9.

$$y = y\big(x(t)\big) = y(e^t)$$

$$\frac{dy}{dt} = \frac{dy}{de^t}\times\frac{de^t}{dt} = \frac{dy}{dx}\times e^t = x\frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right)\frac{dt}{dx} = \frac{1}{\left(\frac{dx}{dt}\right)}\left(-e^{-t}\frac{dy}{dt} + e^{-t}\frac{d^2y}{dt^2}\right) = \frac{1}{x}\left(-\frac{1}{x}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\right)$$

$$\rightarrow \quad x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = 8x \quad \rightarrow \quad \frac{d^2y}{dt^2} - \frac{dy}{dt} + 4\frac{dy}{dt} + 2y = 8e^t \quad \rightarrow \quad \frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 8e^t$$

$$\lambda^2 + 3\lambda + 2 = 0 \quad \rightarrow \quad \lambda = -2, \quad \lambda = -1$$

$$y_b = ce^t, \quad \frac{dy_b}{dt} = ce^t, \quad \frac{d^2y_b}{dt^2} = ce^t, \quad ce^t + 3ce^t + 2ce^t = 6ce^t = 8e^t \quad \rightarrow \quad c = \frac{4}{3}$$

$$\rightarrow \quad y = Ae^{-2t} + Be^{-t} + \frac{4}{3}e^t = \frac{A}{x^2} + \frac{B}{x} + \frac{4x}{3}$$

10.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2, \quad \operatorname{atan}\sqrt{3} = \frac{\pi}{3}$$

$$\int_0^{2a}\int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} r^3\, d\theta\, dr = \int_0^{2a}\left[r^3\theta\right]_{\frac{\pi}{3}}^{\frac{3\pi}{4}} dr = \left(\frac{3\pi}{4} - \frac{\pi}{3}\right)\int_0^{2a} r^3\, dr = \frac{5\pi}{12}\int_0^{2a} r^3\, dr = \frac{5\pi}{12}\left[\frac{r^4}{4}\right]_0^{2a} = \frac{5\pi a^4}{3}$$

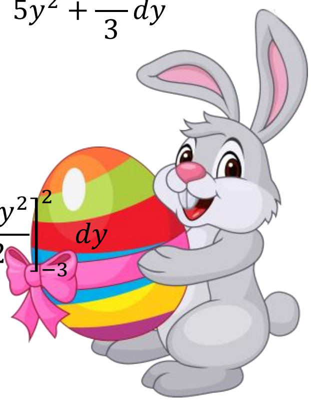
11.

a.

$$M = \int_1^3\int_{-3}^2 \rho(x,y)\,dx\,dy = \int_1^3\int_{-3}^2 x^2 + y^2\,dx\,dy = \int_1^3\left[\frac{x^3}{3} + xy^2\right]_{-3}^2 dy = \int_1^3 5y^2 + \frac{35}{3}dy$$

$$= \left[\frac{5y^3}{3} + \frac{35y}{3}\right]_1^3 = 45 + 35 - \frac{5}{3} - \frac{35}{3} = \frac{200}{3}$$

$$\bar{x} = \frac{1}{M}\int_1^3\int_{-3}^2 x\rho(x,y)\,dx\,dy = \frac{3}{200}\int_1^3\int_{-3}^2 x^3 + xy^2\,dx\,dy = \frac{3}{200}\int_1^3\left[\frac{x^4}{4} + \frac{x^2y^2}{2}\right]_{-3}^2 dy$$



$$= \frac{3}{200}\int_1^3 \left[\frac{x^4}{4}+\frac{x^2y^2}{2}\right]_{-3}^{2} dy = \frac{3}{200}\int_1^3 -\frac{5y^2}{2}-\frac{65}{4}dy = \frac{3}{200}\left[-\frac{5y^3}{6}-\frac{65y}{4}\right]_1^3 = -\frac{3}{200}\times\frac{325}{6}$$

$$\rightarrow \quad \bar{x} = -\frac{13}{16}$$

$$\bar{y} = \frac{1}{M}\int_1^3\int_{-3}^2 y\rho(x,y)dx\,dy = \frac{3}{200}\int_1^3\int_{-3}^2 x^2y+y^3\,dx\,dy = \frac{3}{200}\int_1^3\left[\frac{x^3y}{3}+xy^3\right]_{-3}^2 dy$$

$$\frac{3}{200}\int_1^3 5y^3+\frac{35y}{3}\,dy = \frac{3}{200}\left[\frac{5y^4}{4}+\frac{35y^2}{6}\right]_1^3 = \frac{3}{200}\times\frac{440}{3}$$

$$\rightarrow \quad \bar{y} = \frac{11}{5}$$

b.

$$I_x = \int_1^3\int_{-3}^2 y^2\rho(x,y)dx\,dy = \int_1^3\int_{-3}^2 x^2y^2+y^4dx\,dy = \int_1^3\left[\frac{x^3y^2}{3}+xy^4\right]_{-3}^2 dy$$

$$= \int_1^3 5y^4+\frac{35y^2}{3}dy = \left[y^5+\frac{35y^3}{9}\right]_1^3 = \frac{3088}{9}$$

$$I_y = \int_1^3\int_{-3}^2 x^2\rho(x,y)dx\,dy = \int_1^3\int_{-3}^2 x^4+x^2y^2dx\,dy = \int_1^3\left[\frac{x^5}{5}+\frac{x^3y^2}{3}\right]_{-3}^2 dy$$

$$= \int_1^3 55+\frac{35y^2}{3}dy = \left[55y+\frac{35y^3}{9}\right]_1^3 = \frac{1900}{9}$$

12.

$$\begin{vmatrix}1 & a+b & ab\\ 1 & b+c & bc\\ 1 & c+a & ca\end{vmatrix} = \begin{vmatrix}1 & a+b & ab\\ 0 & c-a & bc-ab\\ 0 & c-b & ca-ab\end{vmatrix} = \begin{vmatrix}c-a & bc-ab\\ c-b & ac-ab\end{vmatrix} = a(c-a)(c-b)-b(c-a)(c-b)$$

$$= (a-b)(c-a)(c-b)$$

13.

a.

$$\begin{vmatrix}3x-7 & -9x+21\\ 3 & -9\end{vmatrix} = -27x+63+27x-63 = 0$$

b.

$$\begin{vmatrix}\cos(2x) & \cos(2x)+\sin(2x)\\ -2\sin(2x) & -2\sin(2x)+2\cos(2x)\end{vmatrix}$$

$$= -2\cos(2x)\sin(2x)+2\cos^2(2x)+2\sin(2x)\cos(2x)+2\sin^2(2x) = 2$$

c.

$$\begin{vmatrix}e^{\alpha x} & e^{\beta x}\\ \alpha e^{\alpha x} & \beta e^{\beta x}\end{vmatrix} = \beta e^{(\alpha+\beta)x} - \alpha e^{(\alpha+\beta)x} = (\beta-\alpha)e^{(\alpha+\beta)x}$$

d.

$$\begin{vmatrix}e^{\alpha x} & xe^{\alpha x}\\ \alpha e^{\alpha x} & (x+\alpha)e^{\alpha x}\end{vmatrix} = (x+\alpha)e^{2\alpha x} - \alpha x e^{2\alpha x} = (x+\alpha-\alpha x)e^{2\alpha x}$$

e.

$$\frac{d}{dx}(x|x|) = |x| + x\frac{d}{dx}|x| = 2x \text{ if } x>0, \quad -2x \text{ if } x<0, \quad \text{undefined if } x=0$$

$$x > 0: \quad \begin{vmatrix} x^2 & x|x| \\ 2x & 2x \end{vmatrix} = 2x^3 - 2x^2|x| = 0$$

$$x < 0: \quad \begin{vmatrix} x^2 & x|x| \\ 2x & -2x \end{vmatrix} = -2x^3 + 2x^2|x| = 0$$

14.

a.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = -2\hat{\imath} + \hat{\jmath} - 3\hat{k} + \hat{\imath} - \hat{\jmath} + \hat{k} = -\hat{\imath} - 2\hat{k} \quad \rightarrow \quad |\overrightarrow{AB}| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \hat{k} + 2\hat{\imath} - \hat{\jmath} + 3\hat{k} = 2\hat{\imath} - \hat{\jmath} + 4\hat{k} \quad \rightarrow \quad |\overrightarrow{BC}| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21}$$

$$\overrightarrow{BA} = -\overrightarrow{AB} = \hat{\imath} + 2\hat{k} \quad \rightarrow \quad \overrightarrow{BA} \cdot \overrightarrow{BC} = 1 \times 2 + 0 \times -1 + 2 \times 4 = 10$$

$$\angle ABC = \text{acos}\left(\frac{10}{\sqrt{5} \times 21}\right) = 0.22$$

b.

$$A_{OAB} = \frac{1}{2}\overrightarrow{OA} \times \overrightarrow{OB} = \frac{1}{2}\left\|\begin{matrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -1 & 1 & -1 \\ -2 & 1 & -3 \end{matrix}\right\| = \frac{1}{2}\left|(-3+1)\hat{\imath} - (3-2)\hat{\jmath} + (-1+2)\hat{k}\right|$$

$$= \frac{1}{2}\sqrt{2^2 + 1^2 + 2^2} = \frac{1}{2}\sqrt{9} = \frac{3}{2}$$

c.

$$\overrightarrow{OA} \times \overrightarrow{OB} = (-3+1)\hat{\imath} - (3-2)\hat{\jmath} + (-1+2)\hat{k} = -2\hat{\imath} - \hat{\jmath} + \hat{k}$$

$$\rightarrow \quad \hat{n}_{OAB} = \frac{\overrightarrow{OA} \times \overrightarrow{OB}}{\sqrt{2^2 + 1^2 + 1^2}} = \frac{-2\hat{\imath} - \hat{\jmath} + \hat{k}}{\sqrt{6}} = -\frac{\sqrt{6}}{3}\hat{\imath} - \frac{\sqrt{6}}{6}\hat{\jmath} + \frac{\sqrt{6}}{6}\hat{k}$$

$$\overrightarrow{OC} \times \overrightarrow{OD} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} \\ -1 & 2 \end{vmatrix} = 2\hat{\imath} + \hat{\jmath}$$

$$\rightarrow \quad \hat{n}_{OCD} = \frac{\overrightarrow{OA} \times \overrightarrow{OB}}{\sqrt{2^2 + 1^2}} = \frac{2\hat{\imath} + \hat{\jmath}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}\hat{\imath} + \frac{\sqrt{5}}{5}\hat{\jmath}$$

d.

$$\boldsymbol{d} \cdot \frac{\boldsymbol{b}}{|\boldsymbol{b}|} = \frac{-1 \times -2 + 2 \times 1 + 0 \times -3}{\sqrt{2^2 + 1^2 + 3^2}} = \frac{2+2}{\sqrt{14}} = \frac{2\sqrt{14}}{7}$$

e.

$$(\boldsymbol{d} \cdot \hat{n}_{OAB}) = -1 \times -\frac{\sqrt{6}}{3} + 2 \times -\frac{\sqrt{6}}{6} = \frac{\sqrt{6}}{3} - \frac{2\sqrt{6}}{6} = 0 \quad \rightarrow \quad \text{orthogonal}$$

$$\boldsymbol{d} - (\boldsymbol{d} \cdot \hat{n}_{OAB})\hat{n}_{OAB} = \boldsymbol{d} = -1\hat{\imath} + 2\hat{\jmath}$$

15.

a.

$$V = \pi R^2 h, \quad V_{\text{Ben}} = \frac{V}{4}, \quad V_{\text{David}} = \frac{V}{2} - V_{\text{Ben}} = \frac{V}{4}$$

$$V_{\text{Dan}} = \int_0^{2\pi}\int_0^R\int_0^{-\frac{hr}{R}\cos\theta} r \, dz \, dr \, d\theta = \int_0^{2\pi}\int_0^R [rz]_0^{-\frac{hr}{R}\cos\theta} dr \, d\theta$$

$$= \int_0^{2\pi} \int_0^R -\frac{hr^2}{R} \cos\theta \, dr \, d\theta = \int_0^{2\pi} \left[-\frac{hr^3}{3R} \cos\theta \right]_0^R d\theta$$

$$= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\frac{hR^2}{3} \cos\theta \, d\theta = \left[-\frac{hR^2}{3} \sin\theta \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{2R^2h}{3} = \frac{2V}{3\pi}$$

$$V_{\text{Malcolm}} = \frac{V}{2} - V_{\text{Dan}} = \frac{V}{2} - \frac{2V}{3\pi} = \left(\frac{1}{2} - \frac{2}{3\pi} \right) V$$

$$\frac{1}{2} - \frac{2}{3\pi} = 0.2878 \ldots \quad \rightarrow \quad V_{\text{Malcolm}} > V_{\text{Ben}} = V_{\text{David}} > V_{\text{Dan}}$$