



1.

a.

$$f(x, y, z) = zx - xy \sin(xyz) + y \cos(xyz) = 0$$

$$\begin{aligned} \left(\frac{\partial x}{\partial y}\right)_z &= -\frac{\left(\frac{\partial f}{\partial y}\right)_{x,z}}{\left(\frac{\partial f}{\partial x}\right)_{y,z}} = -\frac{-x(\sin(xyz) + xyz \cos(xyz)) + \cos(xyz) - xyz \sin(xyz)}{z - y(\sin(xyz) + xyz \cos(xyz)) - y^2z \sin(xyz)} \\ &= \frac{x \sin(xyz) + x^2yz \cos(xyz) - \cos(xyz) + xyz \sin(xyz)}{z - y \sin(xyz) - xy^2z \cos(xyz) - y^2z \sin(xyz)} \end{aligned}$$

$$= \frac{(yz + 1)x \sin(xyz) + (x^2yz - 1) \cos(xyz)}{z - (yz + 1)y \sin(xyz) - xy^2z \cos(xyz)}$$

$$\left(\frac{\partial y}{\partial z}\right)_z = -\frac{\left(\frac{\partial f}{\partial z}\right)_{x,y}}{\left(\frac{\partial f}{\partial y}\right)_{x,z}} = -\frac{x - x^2y^2 \cos(xyz) - xy^2 \sin(xyz)}{-x(\sin(xyz) + xyz \cos(xyz)) + \cos(xyz) - xyz \sin(xyz)}$$

$$= \frac{x - xy^2 \sin(xyz) - x^2y^2 \cos(xyz)}{(yz + 1)x \sin(xyz) + (x^2yz - 1) \cos(xyz)}$$

*Alternative method*

$$zx - xy \sin xyz + y \cos xyz = 0$$

$$\begin{aligned} z \left(\frac{\partial x}{\partial y}\right)_z - \left(\left(\frac{\partial x}{\partial y}\right)_z y + x\right) \sin xyz - xyz \cos xyz \left(\left(\frac{\partial x}{\partial y}\right)_z y + x\right) + \cos xyz \\ - yz \sin xyz \left(\left(\frac{\partial x}{\partial y}\right)_z y + x\right) \end{aligned}$$

$$= \left(\frac{\partial x}{\partial y}\right)_z (z - y \sin xyz - xy^2z \cos xyz - y^2z \sin xyz) + (-x \sin xyz - x^2yz \cos xyz + \cos xyz - xyz \sin xyz)$$

$$\rightarrow \left(\frac{\partial x}{\partial y}\right)_z = \frac{x \sin xyz + x^2yz \cos xyz - \cos xyz + xyz \sin xyz}{z - y \sin xyz - xy^2z \cos xyz - y^2z \sin xyz}$$

$$= \frac{(yz + 1)x \sin xyz + (x^2yz - 1) \cos xyz}{z - (yz + 1)y \sin xyz - xy^2z \cos xyz}$$

$$x - x \left(\frac{\partial y}{\partial z}\right)_x \sin xyz - x^2y \cos xyz \left(\left(\frac{\partial y}{\partial z}\right)_x z + y\right) + \left(\frac{\partial y}{\partial z}\right)_x \cos xyz - yx \sin xyz \left(\left(\frac{\partial y}{\partial z}\right)_x z + y\right)$$

$$= \left(\frac{\partial y}{\partial z}\right)_x (-x \sin xyz - x^2yz \cos xyz + \cos xyz - xyz \sin xyz) + x - x^2y^2 \cos xyz - y^2x \sin xyz$$

$$\rightarrow \left(\frac{\partial y}{\partial z}\right)_x = \frac{x - x^2y^2 \cos xyz - xy^2 \sin xyz}{x \sin xyz + x^2yz \cos xyz - \cos xyz + xyz \sin xyz}$$

$$= \frac{x - xy^2 \sin xyz - x^2y^2 \cos xyz}{(yz + 1)x \sin xyz + (x^2yz - 1) \cos xyz}$$





b.

$$x = r \cosh t \rightarrow \left(\frac{\partial x}{\partial r}\right)_t = \cosh t = \frac{x}{r}, \quad \left(\frac{\partial x}{\partial t}\right)_r = r \sinh t = y$$

$$y = r \sinh t \rightarrow \left(\frac{\partial y}{\partial r}\right)_t = \sinh t = \frac{y}{r}, \quad \left(\frac{\partial y}{\partial t}\right)_r = r \cosh t = x$$

$$\left(\frac{\partial f}{\partial r}\right)_t = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial r}\right)_t + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial r}\right)_t, \quad \left(\frac{\partial f}{\partial t}\right)_r = \left(\frac{\partial f}{\partial x}\right)_y \left(\frac{\partial x}{\partial t}\right)_r + \left(\frac{\partial f}{\partial y}\right)_x \left(\frac{\partial y}{\partial t}\right)_r$$

$$\rightarrow \left(\frac{\partial f}{\partial r}\right)_t = \frac{x}{r} \left(\frac{\partial f}{\partial x}\right)_y + \frac{y}{r} \left(\frac{\partial f}{\partial y}\right)_x, \quad \left(\frac{\partial f}{\partial t}\right)_r = y \left(\frac{\partial f}{\partial x}\right)_y + x \left(\frac{\partial f}{\partial y}\right)_x$$

$$\left(\frac{\partial f}{\partial r}\right)_t^2 - \frac{1}{r^2} \left(\frac{\partial f}{\partial t}\right)_r = \left(\left(\frac{\partial f}{\partial r}\right)_t + \frac{1}{r} \left(\frac{\partial f}{\partial t}\right)_r\right) \left(\left(\frac{\partial f}{\partial r}\right)_t - \frac{1}{r} \left(\frac{\partial f}{\partial t}\right)_r\right)$$

$$= \left(\frac{x}{r} f_x + \frac{y}{r} f_y + \frac{1}{r} (y f_x + x f_y)\right) \left(\frac{x}{r} f_x + \frac{y}{r} f_y - \frac{1}{r} (y f_x + x f_y)\right)$$

$$= \left(\left(\frac{x}{r} + \frac{y}{r}\right) f_x + \left(\frac{x}{r} + \frac{y}{r}\right) f_y\right) \left(\left(\frac{x}{r} - \frac{y}{r}\right) f_x - \left(\frac{x}{r} - \frac{y}{r}\right) f_y\right)$$

$$= \frac{1}{r^2} (x+y)(x-y)(f_x + f_y)(f_x - f_y)$$

$$= \frac{x^2 - y^2}{r^2} (f_x^2 - f_y^2) = \frac{r^2 \cosh^2 t - r^2 \sinh^2 t}{r^2} = r^2 (f_x^2 - f_y^2) = \left(\frac{\partial f}{\partial x}\right)_y^2 - \left(\frac{\partial f}{\partial y}\right)_x^2$$

2.

a.

i.

$$z = -1 - i \rightarrow |z| = \sqrt{2}, \quad |z|^2 = 2 \rightarrow \operatorname{Re}(|z|^2) = 2, \quad \operatorname{Im}(|z|^2) = 0$$

$$\operatorname{Arg}(z) = -\frac{3\pi}{4}, \quad \hat{z} = e^{-\frac{3\pi i}{8}}, \quad u_1 = -1, \quad u_2 = 1$$

$$z_1^{\frac{1}{2}} = -\sqrt{2}^{\frac{1}{2}} e^{-\frac{3\pi i}{8}} = 2^{\frac{1}{4}} \cos\left(\frac{3\pi}{8}\right) - 2^{\frac{1}{4}} i \sin\left(\frac{3\pi}{8}\right), \quad z_2^{\frac{1}{2}} = -2^{\frac{1}{4}} \cos\left(\frac{3\pi}{8}\right) + 2^{\frac{1}{4}} i \sin\left(\frac{3\pi}{8}\right)$$

$$\operatorname{Re}\left(z_1^{\frac{1}{2}}\right) = 2^{\frac{1}{4}} \cos\left(\frac{3\pi}{8}\right), \quad \operatorname{Im}\left(z_1^{\frac{1}{2}}\right) = -2^{\frac{1}{4}} \sin\left(\frac{3\pi}{8}\right)$$

$$\operatorname{Re}\left(z_2^{\frac{1}{2}}\right) = -2^{\frac{1}{4}} \cos\left(\frac{3\pi}{8}\right), \quad \operatorname{Im}\left(z_2^{\frac{1}{2}}\right) = 2^{\frac{1}{4}} \sin\left(\frac{3\pi}{8}\right)$$

ii.

$$||z|^2| = |2| = 2, \quad \operatorname{Arg}(|z|^2) = 0$$

$$z^2 = (-1 - i)(-1 - i) = 1 + 2i - 1 = 2i, \quad |z^2| = |2i| = 2, \quad \operatorname{Arg}(z^2) = \operatorname{Arg}(2i) = \frac{\pi}{2}$$

$$-iz = i - 1, \quad |-iz| = \sqrt{2}, \quad \operatorname{Arg}(-iz) = \frac{3\pi}{4}$$





b.

$$\text{let } z = 1 + \cos 5\theta + i \sin 5\theta, \quad z^* = 1 + \cos 5\theta - i \sin 5\theta$$

$$\frac{z}{z^*} = \frac{|z|e^{i\phi}}{|z|e^{-i\phi}} = e^{2i\phi}$$

$$\tan \phi = \frac{\sin 5\theta}{1 + \cos 5\theta} = \frac{2 \sin \frac{5\theta}{2} \cos \frac{5\theta}{2}}{1 + 2 \cos^2 \frac{5\theta}{2} - 1} = \frac{\sin \frac{5\theta}{2}}{\cos \frac{5\theta}{2}} = \tan \frac{5\theta}{2}$$

$$\rightarrow \phi = \frac{5\theta}{2}, \quad \frac{z}{z^*} = e^{2i\phi} = e^{5i\theta} = \cos 5\theta + i \sin 5\theta$$

3.

a.

$$\arg(z_1) = \text{atan} \frac{1}{2} + 2n\pi, \quad \arg(z_2) = \text{atan} \frac{1}{3} + 2n\pi$$

b.

$$\arg(z_1) + \arg(z_2) = \arg(z_1 z_2) = \arg((2+i)(3+i)) = \arg(5+5i) = \frac{\pi}{4} + 2n\pi$$

$$\rightarrow \frac{\pi}{4} = \text{atan} \frac{1}{2} + \text{atan} \frac{1}{3}$$

c.

$$\frac{d}{dx} \text{atan } x = \frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 - x^6 + \dots$$

$$\rightarrow \text{atan } x = \int 1 - x^2 + x^4 - \frac{x^6}{2} + \dots dx = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\pi \approx 4 \left( \frac{1}{2} - \frac{1}{3} \left( \frac{1}{2} \right)^3 + \frac{1}{3} - \frac{1}{3} \left( \frac{1}{3} \right)^3 \right) = \frac{502}{162} = 3.1173$$

4.

a.

$$\frac{dy}{dx} + \frac{1}{x}y = 1 \rightarrow \text{linear}, \quad \mu = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

$$x \frac{dy}{dx} + y = x, \quad \frac{d}{dx}(xy) = x \rightarrow xy = \int x dx = \frac{x^2}{2} + C \rightarrow y = \frac{x}{2} + \frac{C}{x}$$

b.

$$\frac{dy}{dx} = \frac{x-y}{x} = \frac{f(x,y)}{g(x,y)} = \frac{nx-ny}{nx} = \frac{f(nx,ny)}{g(nx,ny)} \rightarrow \text{homogeneous}$$

$$y = vx \rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx} = 1 - \frac{vx}{x} = 1 - v \rightarrow x \frac{dv}{dx} = 1 - 2v$$

$$\frac{1}{1-2v} dv = \frac{1}{x} dx \rightarrow \int \frac{1}{1-2v} dv = \ln x = \frac{-\ln|1-2v|}{2} = \ln \left| \frac{C}{\sqrt{1-2v}} \right|$$

$$x = \frac{C}{\sqrt{1-2v}} \rightarrow \frac{C}{x^2} = 1 - 2v = 1 - \frac{2y}{x} \rightarrow y = \frac{x}{2} - \frac{C}{2x} = \frac{x}{2} + \frac{C}{x}$$





c.

$$x \frac{dy}{dx} = x - y \rightarrow x dy = (x - y) dx \rightarrow (x - y) dx - x dy = 0$$

$$\left( \frac{\partial(x - y)}{\partial y} \right)_x = -1, \quad \left( \frac{\partial(-x)}{\partial x} \right)_y = -1 \rightarrow \text{exact}$$

$$f(x, y) = \int x - y dx + \psi(y) = \frac{x^2}{2} - yx + \psi(y)$$

$$f(x, y) = \int -x dy + \phi(x) = -xy + \phi(x)$$

$$\rightarrow \phi(x) = \frac{x^2}{2}, \quad \psi(y) = 0, \quad \rightarrow f(x, y) = \frac{x^2}{2} - xy = \int 0 df = C$$

$$xy = \frac{x^2}{2} + C \rightarrow y = \frac{x}{2} + \frac{C}{x}$$

5.

$$\frac{dy}{dx} = -y \tanh x \rightarrow \int -\frac{1}{y} dy = \int \tanh x dx = \int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$$

$$\text{let } u = e^x + e^{-x}, \quad du = e^x - e^{-x} dx$$

$$-\ln y = \int \frac{1}{u} du = \ln u + \ln C = \ln(C \cosh x) \rightarrow y = \frac{C}{\cosh x}$$

6.

$$\frac{dy}{dx} = \frac{z+2}{z-4} = \frac{1}{2} \frac{dz}{dx} - \frac{1}{2} \rightarrow \frac{dz}{dx} = \frac{2z+4}{z-4} + 1 = \frac{2z+4+z-4}{z-4} = \frac{3z}{z-4}$$

$$\frac{dz}{z-4} = \frac{3z}{z-4} \rightarrow x = \int \frac{z-4}{3z} dz = \frac{z}{3} - \frac{4}{3} \ln z + C$$

$$x = \frac{x}{3} + \frac{2y}{3} - \frac{4}{3} \ln(x+2y) + C \rightarrow 2y - 4 \ln(x+2y) - 2x + C = 0$$

7.

$$\frac{d^2u}{ds^2} + 4 \frac{du}{ds} + 5u = 0, \quad \lambda^2 + 4\lambda + 5 = 0 \rightarrow \lambda = -2 \pm \frac{\sqrt{16-20}}{2} = -2 \pm i$$

$$u = Ae^{(-2+i)s} + Be^{(-2-i)s} = Ae^{-2s}(\cos s + i \sin s) + Be^{-2s}(\cos s - i \sin s)$$

$$= e^{-2s}(A+B) \cos s + e^{-2s}(A-B) \sin s = e^{-2s}(C_1 \cos s + C_2 \sin s), \quad C_1, C_2 \in \mathbb{C}$$

$$u(0) = e^{-2 \times 0}(C_1 \cos 0 + C_2 \sin 0) = C_1 = 0$$

$$\frac{du}{ds} = -2e^{-2s}(C_1 \cos s + C_2 \sin s) + e^{-2s}(-C_1 \sin s + C_2 \cos s)$$

$$= e^{-2s}((C_2 - 2C_1) \cos s - (C_1 + 2C_2) \sin s)$$

$$u'(0) = e^{-2 \times 0}((C_2 - 2C_1) \cos 0 - (C_1 + 2C_2) \sin 0) = C_2 - 2C_1 = C_2 = 2$$

$$\rightarrow u(s) = 2e^{-2s} \sin s$$





8.

$$\frac{d^2x}{dt^2} + 9x = 3 - 3\pi t \rightarrow \lambda^2 + 9 = 0 \rightarrow \lambda = \pm 3i$$

$$Ae^{3it} + Be^{-3it} = A \cos 3t + Ai \sin 3t + B \cos 3t - Bi \sin 3t = (A + B) \cos 3t + (A - B)i \sin 3t \\ = C_1 \cos 3t + C_2 \sin 3t$$

$$x_a(t) = C_3t + C_4, \quad x'_a(t) = C_3, \quad x''_a(t) = 0, \quad 9(C_3t + C_4) = 3 - 3\pi t$$

$$\rightarrow C_3 = -\frac{\pi}{3}, \quad C_4 = \frac{1}{3}$$

$$x(t) = C_1 \cos 3t + C_2 \sin 3t - \frac{\pi t}{3} + \frac{1}{3}$$

$$x(0) = C_1 + \frac{1}{3} = \frac{1}{3} \rightarrow C_1 = 0, \quad x'\left(\frac{\pi}{3}\right) = 3C_2 \cos \pi - \frac{\pi^2}{9} = 0 \rightarrow C_2 = \frac{\pi^2}{27}$$

$$x(t) = \frac{\pi^2}{27} \sin 3t - \frac{\pi t}{3} + \frac{1}{3}$$

9.

$$y = y(x(t)) = y(e^t)$$

$$\frac{dy}{dt} = \frac{dy}{de^t} \times \frac{de^t}{dt} = \frac{dy}{dx} \times e^t = x \frac{dy}{dx}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dx} \right) = \frac{d}{dt} \left( e^{-t} \frac{dy}{dt} \right) \frac{dt}{dx} = \frac{1}{\left( \frac{dx}{dt} \right)} \left( -e^{-t} \frac{dy}{dt} + e^{-t} \frac{d^2y}{dt^2} \right) = \frac{1}{x} \left( -\frac{1}{x} \frac{dy}{dt} + \frac{1}{x} \frac{d^2y}{dt^2} \right)$$

$$\rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = 8x \rightarrow \frac{d^2y}{dt^2} - \frac{dy}{dt} + 4 \frac{dy}{dt} + 2y = 8e^t \rightarrow \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} + 2y = 8e^t$$

$$\lambda^2 + 3\lambda + 2 = 0 \rightarrow \lambda = -2, \quad \lambda = -1$$

$$y_b = ce^t, \quad \frac{dy_b}{dt} = ce^t, \quad \frac{d^2y_b}{dt^2} = ce^t, \quad ce^t + 3ce^t + 2ce^t = 6ce^t = 8e^t \rightarrow c = \frac{4}{3}$$

$$\rightarrow y = Ae^{-2t} + Be^{-t} + \frac{4}{3}e^t = \frac{A}{x^2} + \frac{B}{x} + \frac{4x}{3}$$

10.

$$x = r \cos \theta, \quad y = r \sin \theta, \quad x^2 + y^2 = r^2, \quad \text{atan} \sqrt{3} = \frac{\pi}{3}$$

$$\int_0^{2a} \int_{\frac{\pi}{3}}^{\frac{3\pi}{4}} r^3 d\theta dr = \int_0^{2a} [r^3 \theta]_{\frac{\pi}{3}}^{\frac{3\pi}{4}} dr = \left( \frac{3\pi}{4} - \frac{\pi}{3} \right) \int_0^{2a} r^3 dr = \frac{5\pi}{12} \int_0^{2a} r^3 dr = \frac{5\pi}{12} \left[ \frac{r^4}{4} \right]_0^{2a} = \frac{5\pi a^4}{3}$$





11.

a.

$$M = \int_1^3 \int_{-3}^2 \rho(x, y) dx dy = \int_1^3 \int_{-3}^2 x^2 + y^2 dx dy = \int_1^3 \left[ \frac{x^3}{3} + xy^2 \right]_{-3}^2 dy = \int_1^3 5y^2 + \frac{35}{3} dy$$

$$= \left[ \frac{5y^3}{3} + \frac{35y}{3} \right]_1^3 = 45 + 35 - \frac{5}{3} - \frac{35}{3} = \frac{200}{3}$$

$$\bar{x} = \frac{1}{M} \int_1^3 \int_{-3}^2 x \rho(x, y) dx dy = \frac{3}{200} \int_1^3 \int_{-3}^2 x^3 + xy^2 dx dy = \frac{3}{200} \int_1^3 \left[ \frac{x^4}{4} + \frac{x^2 y^2}{2} \right]_{-3}^2 dy$$

$$= \frac{3}{200} \int_1^3 \left[ \frac{x^4}{4} + \frac{x^2 y^2}{2} \right]_{-3}^2 dy = \frac{3}{200} \int_1^3 -\frac{5y^2}{2} - \frac{65}{4} dy = \frac{3}{200} \left[ -\frac{5y^3}{6} - \frac{65y}{4} \right]_1^3 = -\frac{3}{200} \times \frac{325}{6}$$

$$\rightarrow \bar{x} = -\frac{13}{16}$$

$$\bar{y} = \frac{1}{M} \int_1^3 \int_{-3}^2 y \rho(x, y) dx dy = \frac{3}{200} \int_1^3 \int_{-3}^2 x^2 y + y^3 dx dy = \frac{3}{200} \int_1^3 \left[ \frac{x^3 y}{3} + xy^3 \right]_{-3}^2 dy$$

$$\frac{3}{200} \int_1^3 5y^3 + \frac{35y}{3} dy = \frac{3}{200} \left[ \frac{5y^4}{4} + \frac{35y^2}{6} \right]_1^3 = \frac{3}{200} \times \frac{440}{3}$$

$$\rightarrow \bar{y} = \frac{11}{5}$$

b.

$$I_x = \int_1^3 \int_{-3}^2 y^2 \rho(x, y) dx dy = \int_1^3 \int_{-3}^2 x^2 y^2 + y^4 dx dy = \int_1^3 \left[ \frac{x^3 y^2}{3} + xy^4 \right]_{-3}^2 dy$$

$$= \int_1^3 5y^4 + \frac{35y^2}{3} dy = \left[ y^5 + \frac{35y^3}{9} \right]_1^3 = \frac{3088}{9}$$

$$I_y = \int_1^3 \int_{-3}^2 x^2 \rho(x, y) dx dy = \int_1^3 \int_{-3}^2 x^4 + x^2 y^2 dx dy = \int_1^3 \left[ \frac{x^5}{5} + \frac{x^3 y^2}{3} \right]_{-3}^2 dy$$

$$= \int_1^3 55 + \frac{35y^2}{3} dy = \left[ 55y + \frac{35y^3}{9} \right]_1^3 = \frac{1900}{9}$$

12.

$$\begin{vmatrix} 1 & a+b & ab \\ 1 & b+c & bc \\ 1 & c+a & ca \end{vmatrix} = \begin{vmatrix} 1 & a+b & ab \\ 0 & c-a & bc-ab \\ 0 & c-b & ca-ab \end{vmatrix} = \begin{vmatrix} c-a & bc-ab \\ c-b & ac-ab \end{vmatrix} = a(c-a)(c-b) - b(c-a)(c-b)$$

$$= (a-b)(c-a)(c-b)$$





13.

a.

$$\begin{vmatrix} 3x-7 & -9x+21 \\ 3 & -9 \end{vmatrix} = -27x + 63 + 27x - 63 = 0$$

b.

$$\begin{vmatrix} \cos(2x) & \cos(2x) + \sin(2x) \\ -2\sin(2x) & -2\sin(2x) + 2\cos(2x) \end{vmatrix} \\ = -2\cos(2x)\sin(2x) + 2\cos^2(2x) + 2\sin(2x)\cos(2x) + 2\sin^2(2x) = 2$$

c.

$$\begin{vmatrix} e^{\alpha x} & e^{\beta x} \\ \alpha e^{\alpha x} & \beta e^{\beta x} \end{vmatrix} = \beta e^{(\alpha+\beta)x} - \alpha e^{(\alpha+\beta)x} = (\beta - \alpha)e^{(\alpha+\beta)x}$$

d.

$$\begin{vmatrix} e^{\alpha x} & xe^{\alpha x} \\ \alpha e^{\alpha x} & (x+\alpha)e^{\alpha x} \end{vmatrix} = (x+\alpha)e^{2\alpha x} - \alpha xe^{2\alpha x} = (x+\alpha - \alpha x)e^{2\alpha x}$$

e.

$$\frac{d}{dx}(x|x|) = |x| + x \frac{d}{dx}|x| = 2x \text{ if } x > 0, \quad -2x \text{ if } x < 0, \quad \text{undefined if } x = 0$$

$$x > 0: \begin{vmatrix} x^2 & x|x| \\ 2x & 2x \end{vmatrix} = 2x^3 - 2x^2|x| = 0$$

$$x < 0: \begin{vmatrix} x^2 & x|x| \\ 2x & -2x \end{vmatrix} = -2x^3 + 2x^2|x| = 0$$

14.

a.

$$\overline{AB} = \overline{OB} - \overline{OA} = -2\hat{i} + \hat{j} - 3\hat{k} + \hat{i} - \hat{j} + \hat{k} = -\hat{i} - 2\hat{k} \rightarrow |\overline{AB}| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\overline{BC} = \overline{OC} - \overline{OB} = \hat{k} + 2\hat{i} - \hat{j} + 3\hat{k} = 2\hat{i} - \hat{j} + 4\hat{k} \rightarrow |\overline{BC}| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21}$$

$$\overline{BA} = -\overline{AB} = \hat{i} + 2\hat{k} \rightarrow \overline{BA} \cdot \overline{BC} = 1 \times 2 + 0 \times -1 + 2 \times 4 = 10$$

$$\angle ABC = \arccos\left(\frac{10}{\sqrt{5} \times \sqrt{21}}\right) = 0.22$$

b.

$$A_{OAB} = \frac{1}{2} \overline{OA} \times \overline{OB} = \frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -1 \\ -2 & 1 & -3 \end{vmatrix} = \frac{1}{2} [(-3+1)\hat{i} - (3-2)\hat{j} + (-1+2)\hat{k}]$$

$$= \frac{1}{2} \sqrt{2^2 + 1^2 + 2^2} = \frac{1}{2} \sqrt{9} = \frac{3}{2}$$

c.

$$\overline{OA} \times \overline{OB} = (-3+1)\hat{i} - (3-2)\hat{j} + (-1+2)\hat{k} = -2\hat{i} - \hat{j} + \hat{k}$$

$$\rightarrow \hat{n}_{OAB} = \frac{\overline{OA} \times \overline{OB}}{\sqrt{2^2 + 1^2 + 1^2}} = \frac{-2\hat{i} - \hat{j} + \hat{k}}{\sqrt{6}} = -\frac{\sqrt{6}}{3}\hat{i} - \frac{\sqrt{6}}{6}\hat{j} + \frac{\sqrt{6}}{6}\hat{k}$$

$$\overline{OC} \times \overline{OD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} \\ -1 & 2 \end{vmatrix} = 2\hat{i} + \hat{j}$$

$$\rightarrow \hat{n}_{OCD} = \frac{\overline{OC} \times \overline{OD}}{\sqrt{2^2 + 1^2}} = \frac{2\hat{i} + \hat{j}}{\sqrt{5}} = \frac{2\sqrt{5}}{5}\hat{i} + \frac{\sqrt{5}}{5}\hat{j}$$





d.

$$\mathbf{d} \cdot \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{-1 \times -2 + 2 \times 1 + 0 \times -3}{\sqrt{2^2 + 1^2 + 3^2}} = \frac{2 + 2}{\sqrt{14}} = \frac{2\sqrt{14}}{7}$$

e.

$$(\mathbf{d} \cdot \hat{\mathbf{n}}_{OAB}) = -1 \times -\frac{\sqrt{6}}{3} + 2 \times -\frac{\sqrt{6}}{6} = \frac{\sqrt{6}}{3} - \frac{2\sqrt{6}}{6} = 0 \rightarrow \text{orthogonal}$$

$$\mathbf{d} - (\mathbf{d} \cdot \hat{\mathbf{n}}_{OAB})\hat{\mathbf{n}}_{OAB} = \mathbf{d} = -1\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$$

15.

a.

$$V = \pi R^2 h, \quad V_{\text{Ben}} = \frac{V}{4}, \quad V_{\text{David}} = \frac{V}{2} - V_{\text{Ben}} = \frac{V}{4}$$

$$\begin{aligned} V_{\text{Dan}} &= \int_0^{2\pi} \int_0^R \int_0^{-\frac{hr}{R} \cos \theta} r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^R [rz]_0^{-\frac{hr}{R} \cos \theta} \, dr \, d\theta \\ &= \int_0^{2\pi} \int_0^R -\frac{hr^2}{R} \cos \theta \, dr \, d\theta = \int_0^{2\pi} \left[ -\frac{hr^3}{3R} \cos \theta \right]_0^R \, d\theta \\ &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} -\frac{hR^2}{3} \cos \theta \, d\theta = \left[ -\frac{hR^2}{3} \sin \theta \right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = \frac{2R^2 h}{3} = \frac{2V}{3\pi} \end{aligned}$$

$$V_{\text{Malcolm}} = \frac{V}{2} - V_{\text{Dan}} = \frac{V}{2} - \frac{2V}{3\pi} = \left( \frac{1}{2} - \frac{2}{3\pi} \right) V$$

$$\frac{1}{2} - \frac{2}{3\pi} = 0.2878 \dots \rightarrow V_{\text{Malcolm}} > V_{\text{Ben}} = V_{\text{David}} > V_{\text{Dan}}$$

